Boards and Potentials for Math Modeling in Elementary School

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One of the most important math competences is the ability to use and build mathematical models. This ability is included in the PISA Framework 2003 (OECD, 2003) and the Common Core State Standards for Mathematics. Several of the important models widely used by experts are based on potential theory. We illustrate here how to adapt these ideas for use from first grade. We have been using these types of models for several years with thousands of students using concrete boards and online systems. We followed two strategies (Araya, 2012). The USABle strategy: first **u**se a model, then **s**elect models, then **a**djust models and finally **b**uild models. The other is the CCS strategy: start with **c**oncrete models, then **c**omputer models, and then **s**ymbolic models.

Introduction

Why model? Building models is a basic thinking activity (Epstein, 2008). We are continuously imagining landscapes, situations, relations, and using them to run simulations in our heads in order to make decisions and act. Most of these models are implicit. In mathematical modeling we try to make them as explicit as possible, specifying its main components and the relations between them. For example, if you are planning to build a bridge, you start by imagining different possible bridges and the traffic across them. Normally this is an implicit process, and you do it automatically. If you have more time and if you are in charge of designing it, then you try to imagine its components and how it will respond to traffic load, water flow and wind. You start a more deliberate and explicit process. You can build a small toy bridge, test it, and measure its response to different stresses. Then you have to think how to scale it. Here it is critical that you try to determine the relations between the different components, testing them on your toy model and eventually on a more realistic toy model. This whole process from a first implicit model to an explicit one with mathematical specifications is the mathematical model building process that allows us to make better decisions. It is a key ability that we need our students to learn and start using systematically in their daily lives.

However, to build models requires experience in specific fields and in different ways of thinking about how to capture relations between components. Normally experts on mathematical models have been working several years on math modeling in a particular field. Only once they have enough experience are they capable of making improvements on the models in their field. Very rarely will one expert propose a radically new math model. Therefore we cannot expect that our students will propose math models from nowhere.

One typical and powerful way of thinking is potential theory, where forces and motion come from an intensity on a board. You imagine the world as a board. For example, a landscape is a rectangular board, like a chess board, but in each cell you have a number that represents the height of the landscape above the cell. You can also imagine the number on each cell as the amount of food on the cell, or the intensity of the odor on the cell. According to the mathematician John Holland (Holland, 1998), for science and mathematical modeling, boards are as basic and fundamentals as numbers. Somehow our K12 educational system hasn't recognized it. In the curriculum we only have the number, geometry, algebra and statistics strands, but not a board strand. This is a pity, not only because boards are a basic component of mathematics modeling, but they are a very engaging and stimulating way of thinking. Particularly for young kids, they are really fun. In this paper we will show different math models using boards and a strategy to develop a way of thinking and modeling with potentials.

The USABle Strategy

In order to design strategies to disseminate Mathematical Modeling skills in the K-12 school system, we need to understand how scientists, engineers and other professionals do mathematical modeling. Some people have built models from zero, but this is very uncommon. Normally, experts have been working for several years on existing types of models. So we need to know the main types of mathematical models that they use. Also there are certain patterns and kinds of techniques that experts use. We need to identify them and then teach that to our teachers.

Following what experts do, we have proposed a strategy to teach mathematical modeling. It has four stages (Araya, 2012). The first stage is to make students **u**se certain types of typical and fertile (Epstein, 2008) mathematical models; the student learns certain ways of thinking with models that can be used in a great diversity of situations. Once the student knows how to use a model and has used it several times, then it starts a second stage where in a given situation s/he has to **s**elect the most appropriate model from a set of two to five options. In the third stage, s/he has to **a**djust parameters in a model in order to best fit a situation. In the fourth stage, s/he has to **b**uild a rather new model or generalize a known model for a new situation. We have named this strategy the **USAB**le strategy (Araya, 2012; see also Lingefjard, 2007).

It is very important that the models introduced are really very fertile and that at consecutively higher grades they can progressively be augmented to include more details and complex phenomena. In this way, students will every year review the same type of models but with more sophisticated mathematics and more parameters in order to account for more realistic situations. This strategy will ensure that they incorporate particular strategies and ways of thinking that can be used in building important and widely used types of mathematical models.

What trajectory will my pill bug follow?

Imagine you have to analyze the trajectories that will follow bacteria, worms, pill bugs or any other simple organism. You can start by placing a pill bug on a slide projector or in front of a video camera, and project the organism on a white blackboard. If a student marks on the whiteboard the place of the projected pill bug she will obtain a trajectory. In this case, she can see that a trajectory is a sequence of positions. You can have several students on the whiteboard marking the positions of different projected pill bugs, each one with markers of different colors.

Now the question is to try to imagine a mechanism that follows these organisms to decide where to move (Loeb, 1918). The typical basic mechanism requires that you think in time steps. At each moment, think where it will go in the next instant. The second imagination device is that you think of the space as a board, and that in each cell of the board there is a number indicating the amount of food on the cell. The third component is the movement decision mechanism: at each moment the pill bug looks at its eight neighbor cells and selects to move to the cell with more food, but only if that cell has more food than the cell where it is now. Figures 1a, b and c are screen shots of a typical activity for first graders.

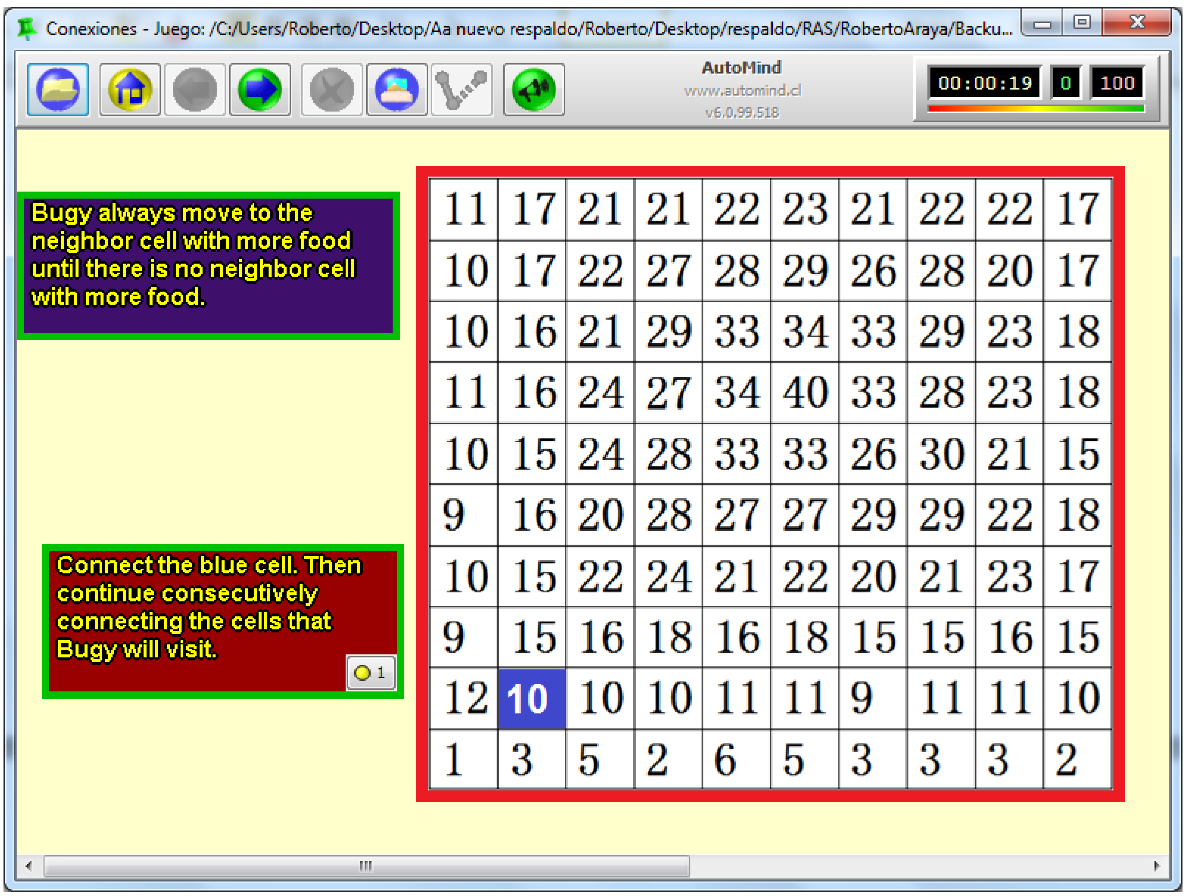

Figure 1a: Screenshot of the problem to find the trajectory.

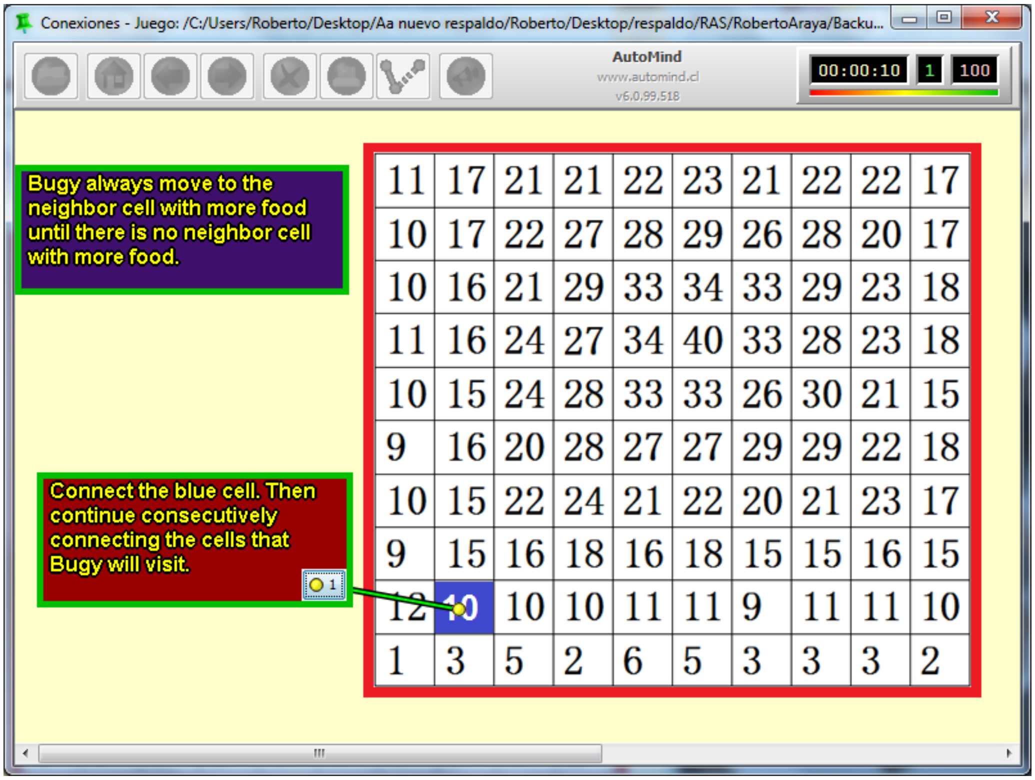

Figure 1b: Screenshot after the student connects the initial cell.

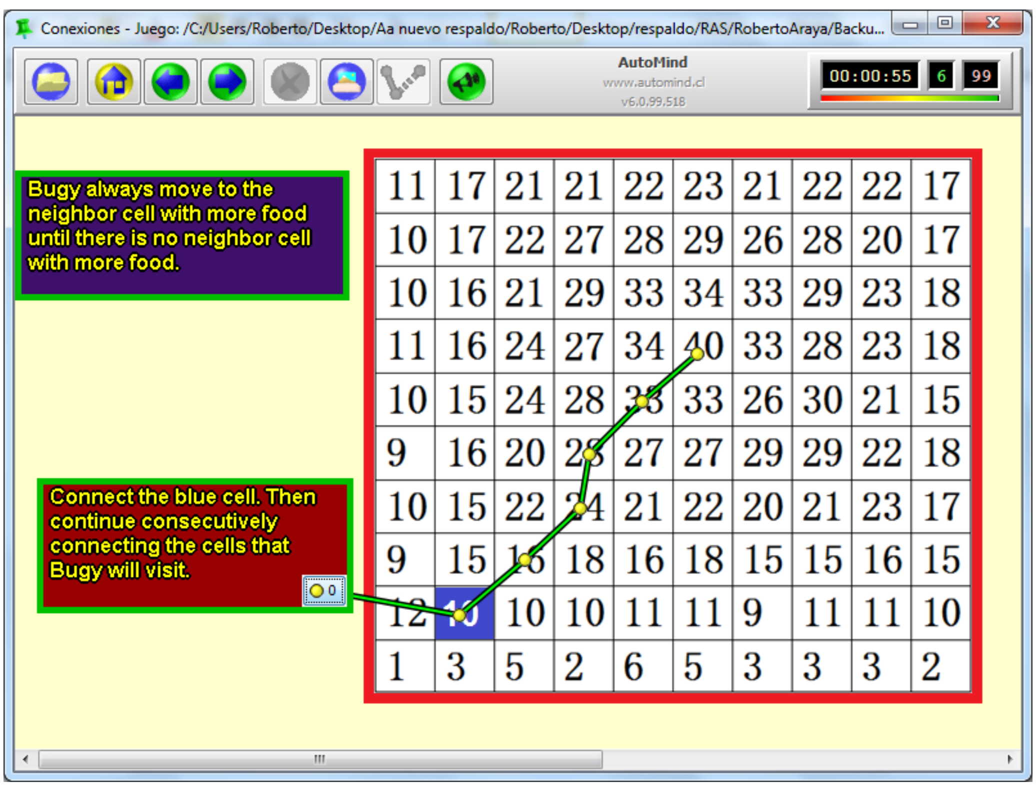

Figure 1c: Screenshot after the student has consecutively connected all the cells according to the rule of motion.

Here the student has to follow the rule of motion and has to compare eight pairs of integers. This way the student will generate a trajectory. This is the first stage of the USABle strategy: use a model. According to our experience with thousands of students, this activity is very attractive for them. It is much more interesting than to just have a list of pairs of numbers and decide in each case which number is bigger. Furthermore, the teacher can easily tell stories about an organism or several organisms that are moving to better places with more food. For example, which organism will first reach the better position on the board. We are storytelling machines. Humans, and particularly kids, really enjoy hearing and making up stories. Stories not only engage them, but also connect them with a meaningful social world. This is an excellent didactic opportunity to engage students. This activity can be started on a tiled floor. Each tile is given a number and students must follow the rules of motion in order to determine how to move.

After using this model with different boards, we can go to the next stage: select a model. For example, we can present two boards and ask which one represents a situation with only one place with more food. Another exercise is to select the board where whenever you start you get to the place with the most food. Then we go to the third stage, where you ask students to adjust a parameter in the model in order to better represent a situation. For example, to change a number on the board to obtain a certain feature. This could be a feature like the following: if an organism starts in a given position then it can reach the cell with the most food. Or it could be a feature like the following: two organisms starting at some given positions will eventually meet.

Will Tina´s dog find the hidden bone?

Tina´s dog is searching for a bone that she has hidden somewhere in the park. The dog starts sniffing and moving according to the odor intensity. Will her dog find the bone?

Here the students have to figure out how to model this new situation. They have to imagine the park as a board. However, now the numbers on the cells are the intensity of the odor, not the amount of food. The bone generates the maximum odor intensity in the location where it is hidden. As before, they have to imagine a rule of motion. They have to think again in terms of time steps and that the dog will move in the next moment to the neighbor cell with the highest odor intensity, but only if this intensity is higher than the one where the dog is right now. Figure 2 shows a screenshot of this activity.

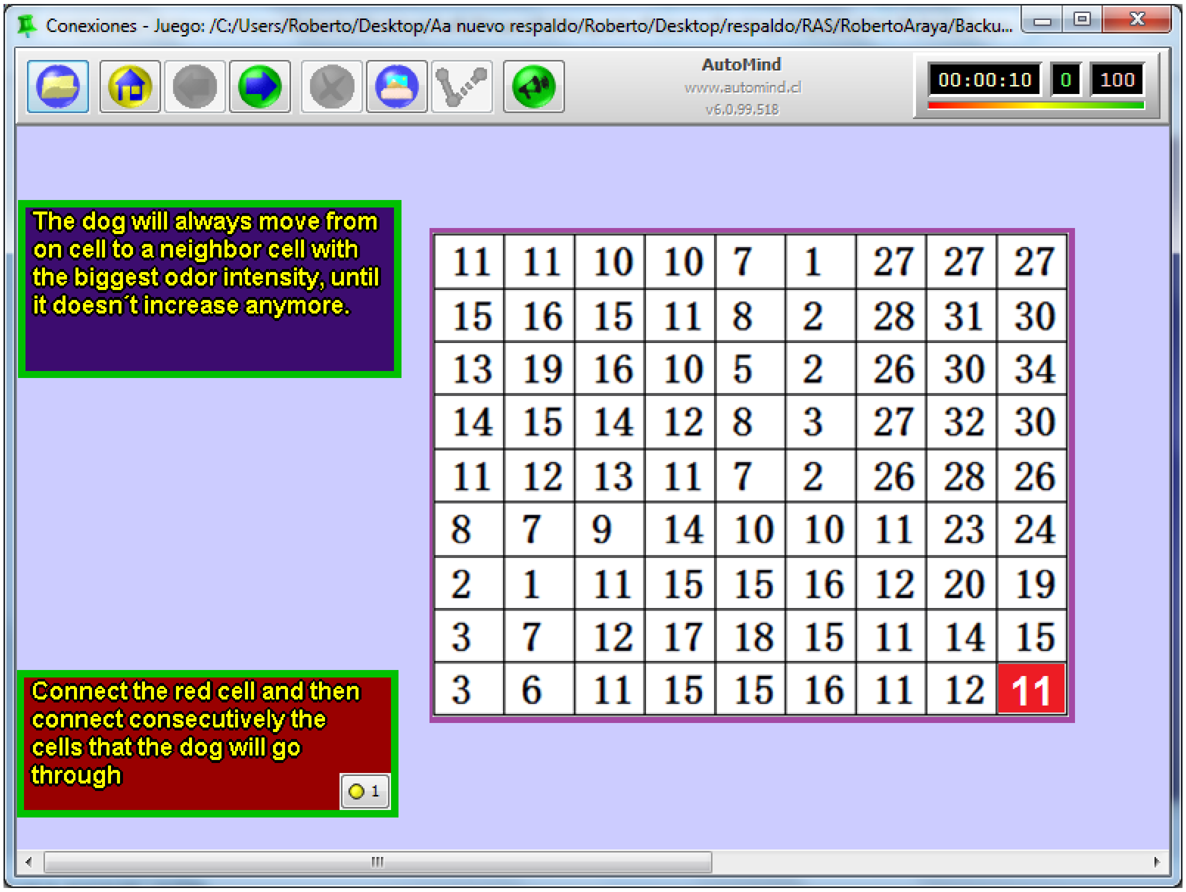

Figure 2: Screenshot of the activity to find out the trajectory of a dog searching for the hidden bone

Here again we propose that the student go through the four stages. After using this model, she can select a board that best represents situations where there are two hidden bones in two separate locations, or a situation where a dog could get stuck without moving and finding a bone. Then she can go to the next stage where she can change a number on the board to represent the situation where starting from different positions two dogs could never meet. Then the student can think about how to generalize the model in order to have a more fine grained representation of the park, or what would happen if there is some wind.

This type of model can also easily include higher numbers appropriate for second and third graders, or other domains like decimals and fractions appropriate for fourth, fifth and sixth graders. The boards can include powers and radicals as well.

Will this rock crash my home and kill my teddy bear?

There is a rock on the higher point of a hill, and during a typhoon or hurricane the alarm was given for a high probability that the rock will fall down. Jane and her family have abandoned their house, and they are now safe in a government provided shelter. Then Jane realizes that she had forgotten to bring her teddy bear. Will the rock crash her house and kill her teddy bear?

This is a mathematical modeling problem that allows us to pose a more demanding challenge. The world is again a board, but now the numbers of the cells represent something of a different nature. Now the student has to reflect and consider that the numbers are not amount of food or odor. They have to discover that here numbers are heights. However there is a much more critical change: now the student has to change the rule of motion. The rock will move to the neighbor cell with lowest number, not the highest number. This is exactly opposite to what they have done before. Besides that, this problem allows us to pose an emotionally charged story, one that is very real and meaningful for kids. Teddy bears, also called transitional objects by psychologists, involve powerful affective states. They are a great source of motivation, and kids will be very interested in accepting the challenge and work hard to find out what will happen at the end of the story.

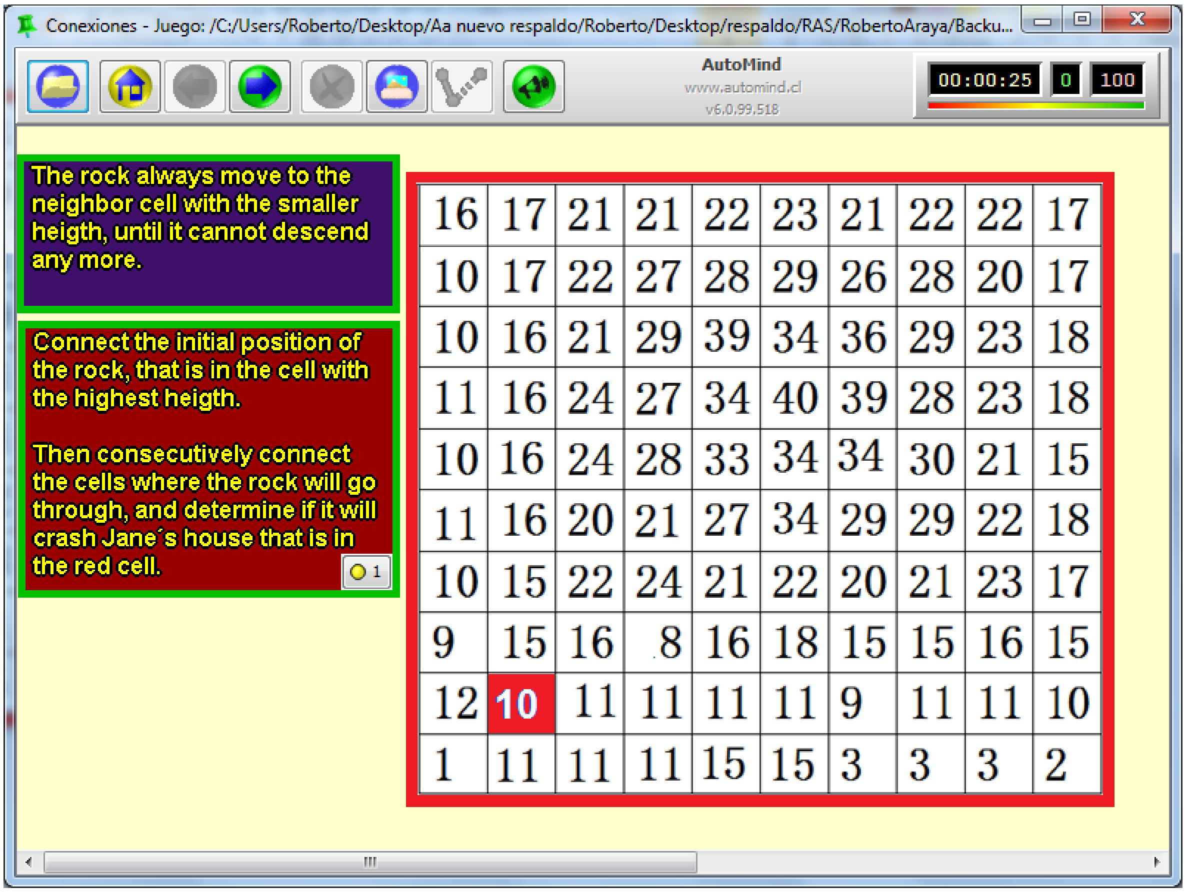

Figure 3: Screenshot of the activity to find out if the rock will crash Jane´s house

Rules and patterns

Consider now a board whose numbers are filled with a rule or pattern. In this case the numbers on the cells are not explicitly given as before. The student has to interpret a rule and fill the numbers. For example, a board where all rows are identical, all of them have a zero in the left extreme, and every cell is one unit greater than its left neighbor. Can you draw the board and fill the numbers? Will the rock crash Jane´s home and kill her lovely teddy bear? Under what conditions will the rock not crash Jane´s home? Do you need to do all the comparisons and connections, or can you just decide immediately? Why? Using cardboard, can you build a three dimensional model of the hill for this case?

There are several other rules you can try. Another simple rule is the one where all columns are identical. All of them have a zero in the bottom extreme, and every cell is one unit greater than the cell just below it. Can you draw the board and fill the numbers? Will the rock crash Jane´s home and kill her teddy bear? Under what conditions won't the rock crash Jane´s home?

If there is some kind of pattern that is much more efficient to describe the landscape with rules instead of numbers, it has much less information that has to be saved on the computers and the computations are much faster. What other type of patterns can be described by simple rules?

Another rule comes from adding the numbers in each cell from the two previous boards. Can you draw the board and fill the numbers? Will the rock crash Jane´s home and kill her teddy bear? Under what conditions won't the rock crash Jane´s home? Do you need to do all the comparisons and connections or can you just decide immediately? Why?

By adding and subtracting boards your students can generate new boards with interesting patterns. You can also try multiplying every cell of a board by two. For example, consider the first board described in this section. Multiply by two every cell of this board. Can you draw the board and fill the numbers on each cell? Will the rock now crash Jane´s home and kill her teddy bear? Under what conditions won't the rock crash Jane´s home? What happens if you multiply every cell by three instead of two?

Interestingly, with this examples not only can you have your students engaged doing additions, subtractions and multiplications besides number comparisons, but you also have introduced them to matrices, addition between matrices and multiplication of matrices by a scalar. This provides a great entrance to a symbolic world (see Araya et Al, 2012, for another strategy through decision games).

A spatial metaphor to make your own chocolate

Imagining the world as a board and imagining motion as a product of a very simple mechanism that involves a potential function is a very powerful and fertile way of thinking. This way of thinking goes all the way to level curves on maps, gravity potential and explaining the motion of stars and planets, electromagnetic potentials, isobar curves and winds. The spatial board can also be used to model other situations that seem very far from landscapes with rocks and animal movements.

Mary and John are making chocolate and are exploring how much milk and sugar they should include. They want to be as professional as possible and make the best chocolate. They start with a known recipe obtained from a web page that specifies a quantity of milk and some other quantity of sugar. Then they explore eight options to modify the recipe. Option one is adding one unit of milk; option two 2 is adding one unit of sugar; option three is with one unit less of milk; option four is with one unit less of sugar; option five is adding both one unit of sugar and one unit of milk; option six is adding one unit of sugar but with one unit less of milk; option seven is with one unit less of sugar but adding one unit of milk; and finally option eight is with one unit less of sugar and one unit less of milk. For the initial recipe and for each of these eight options, Mary and John have their classmates taste the chocolate and score how good it is. They average their classmates' scores and get the final score for the original mix and for each of the eight options. They select the best option, and then continue the whole process again with eight new options. They continue this process until no improvement is obtained in the scores.

Now comes a little thinking. Does this whole thing seem somehow similar to the previous situations of organism tropism or rock movement? You can imagine that you have again a board with rows and columns, but instead of locations they correspond to amounts of sugar and milk. For example, the cell in the third row and fifth column corresponds to a combination with three units of sugar and five units of milk, and the number on the cell is not the height but the average of their classmate's scores when testing the chocolate with those combinations of sugar and milk. The search mechanism of testing the eight options is not an organism´s search to move, but a search to change the mixture of ingredients in order to climb a mountain of savors. After some steps repeating this process a cell will be obtained with a high score. In this cell, the search of all eight options will generate a worse score than the reached position. This is a great mixture of sugar and milk to make the chocolates. Mary and John´s classmates will love it.

In this section we have not introduced a spatial activity as in the previous examples, but a spatial metaphor. Finding the mixture with maximum savor is similar to climbing a hill in a foggy day. You don´t know beforehand the numbers on every cell. Each time you want to move from a cell, you have to do some tests (that will take time and money), and then you will know the numbers but only in the neighbor cells. In this activity, students are initiated in the widely used optimization methods of linear and nonlinear programing. They are also including statistics together with number comparison, boards, and potentials.

Biking skills

Tom is biking on the school patio. The floor is flat. There is no slope nor bumps. The tiled floor is so clean and slippery that there is practically no friction. Once Tom gets some speed, his bicycle continues to move indefinitely even when he's not pedaling. Tom is in the extreme left of the bottom row of the board as shown in figure 4a, and he starts with a velocity of two horizontal cells per unit time. Will he hit the blue ball if he doesn´t pedal at all? If he has one opportunity to pedal and move one position up, where should he do it in order to hit the blue ball?

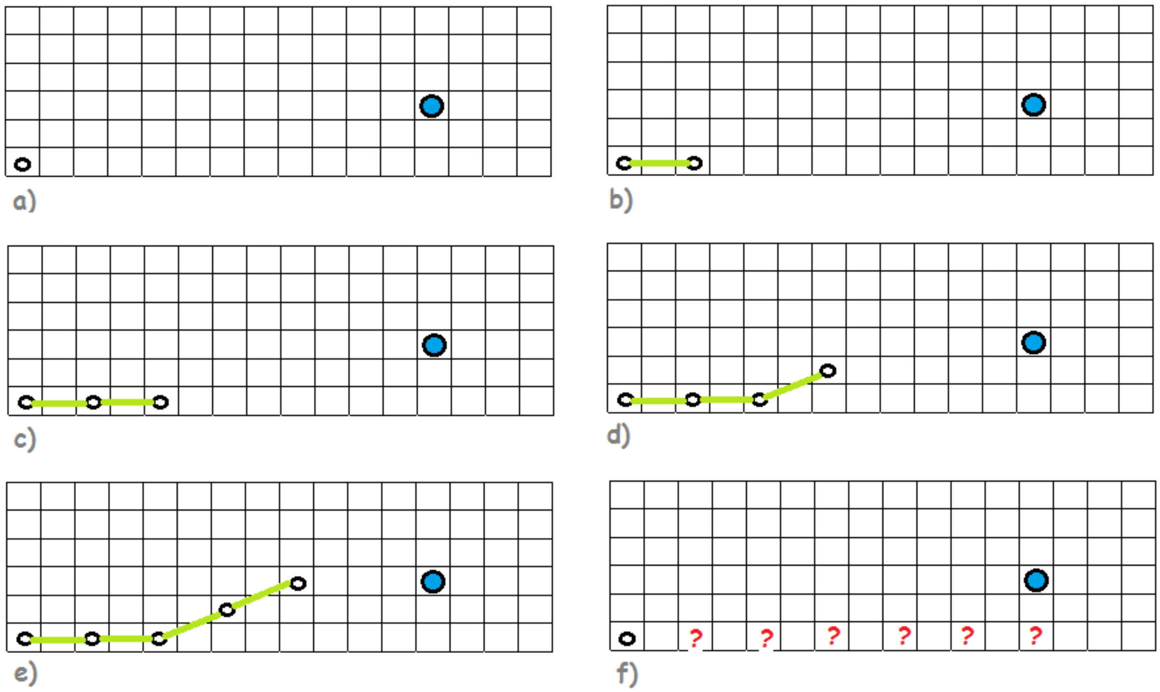

Figure 4: Different time frames with associated trajectory when Tom is riding his bike.

In this case we can imagine the patio as a board completely filled with zeroes. Then, besides the motion from the initial velocity, all other motion comes from the force that Tom himself pushes with his legs. This means that if he doesn´t pedal, then in the next moment he will reach the third cell as shown in figure 4b. If he again doesn´t pedal then in the next moment he will reach the fifth cell from the extreme left as show in figure 4c. This way he will continue to go to the right, advancing two cells from one moment to the next. We say that he is moving according to an inertia vector that is two cells to the right.

What will happen if in the fifth cell from left to right he pedals one unit in the upper direction? Let´s assume that one unit of pedaling gives him one unit of velocity. What cell will he reach at the next moment? Here you have to consider the composition of two movements. One movement is two positions to the right due to inertia, and then another movement one position up due to the pedaling. This is shown in figure 4d.

What will happen then if he doesn´t pedal anymore? What cell will he reach in the next moment? You have to consider that Tom now has an inertia vector that is two positions to the right and one position up. If he doesn´t do anything, then he will move according to this movement and reach the position indicated in figure 4e. What cell will he reach in the next moment? Given that he doesn´t pedal anymore, he again will move two positions to the right and one position up.

Now let´s consider the original question. If Tom has one opportunity to pedal and move one position up, where should he do it in order to hit the ball? Select one of the red question marks in figure 4g.

If there is no friction, will the rock crash my house and kill my teddy bear?

Now that we know that there is inertia, we can review Jane´s problem and consider this effect to find out if the rock will crash her house and kill her teddy bear. Oh my god! Maybe inertia is bad news. The movement of the rock is the effect of two movements. One is due to inertia. The rock will continue with the same velocity as before. This means that you have to compute how many cells the rock advanced horizontally and vertically last time and now move the rock in the same way. Besides that you have to consider the direction of higher descent at the cell where the rock was last time. Now after the movement from inertia you move the rock in that direction.

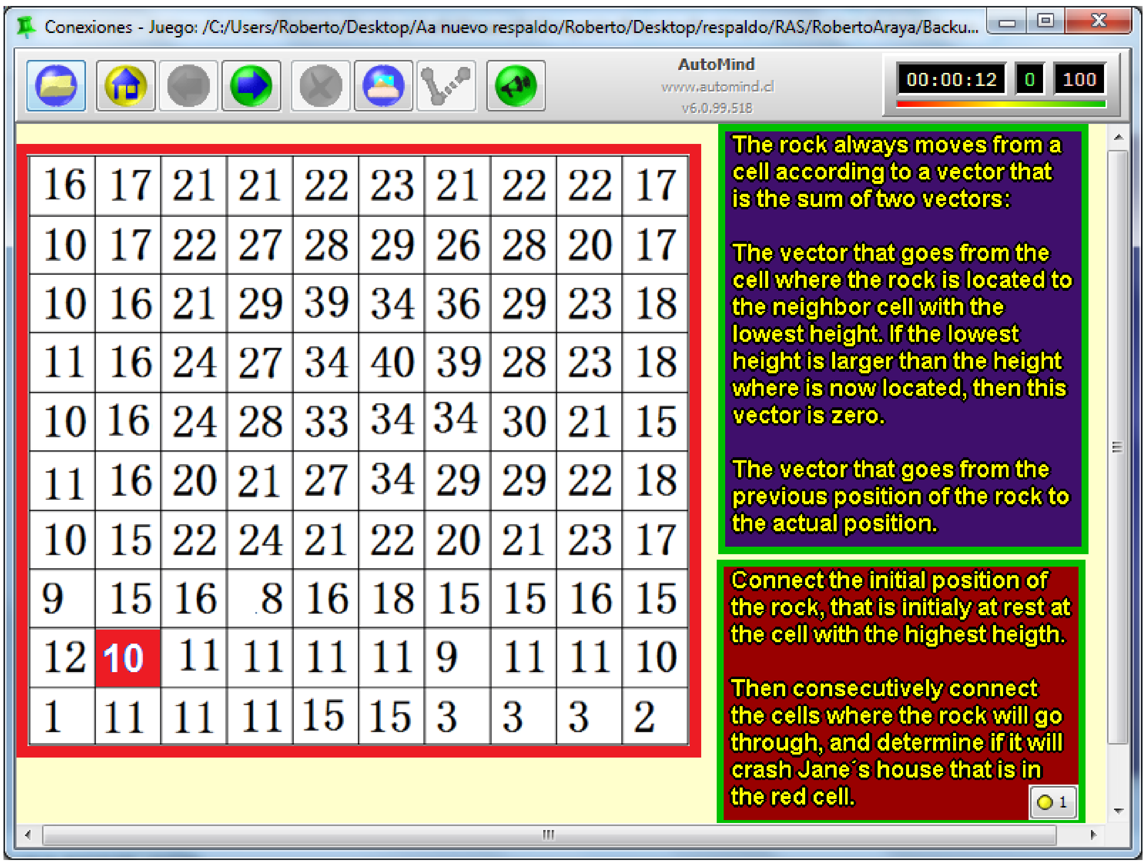

Figure 5: Screenshot of the activity to find out if the rock will crash Jane´s house, in the case of movement with inertia

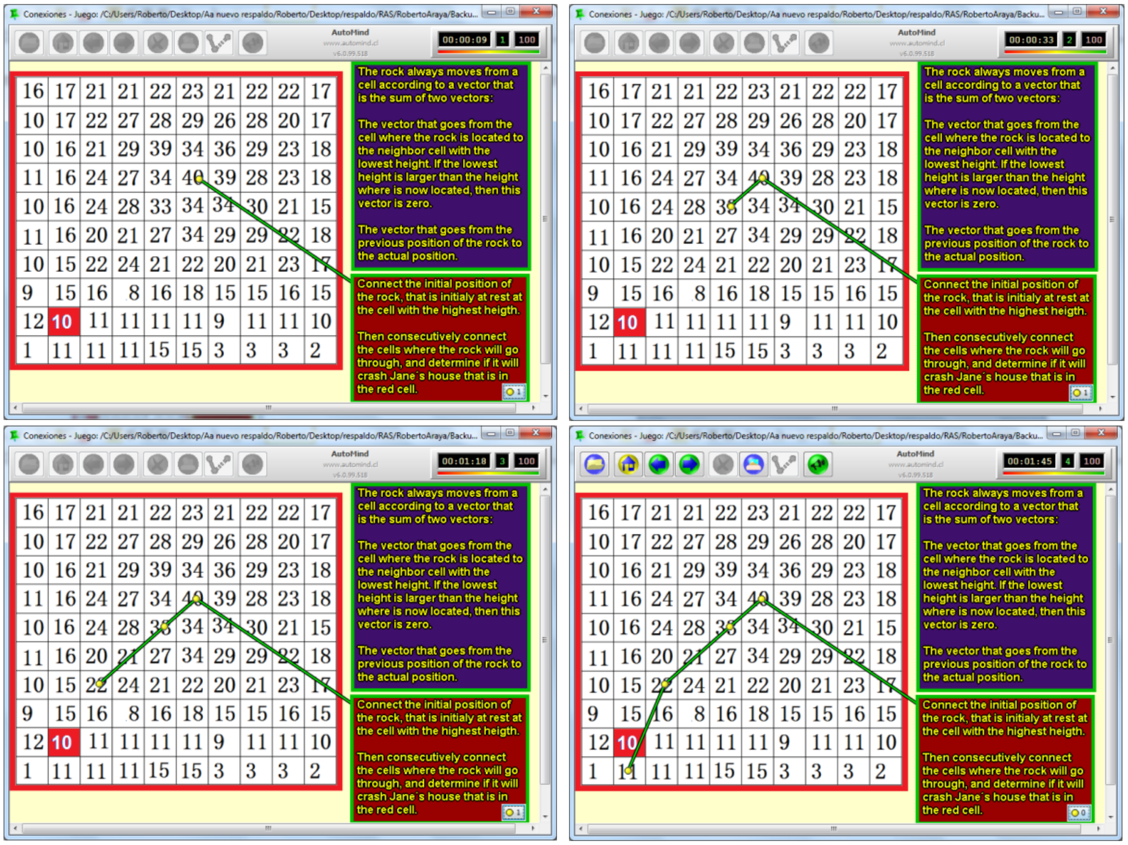

Figure 6: Screenshot of the consecutive 4 steps to solve the activity

A better model would be one where the force (and correspondent movement) that pushes the rock will be proportional to the fall between cells. If the drop is higher, like on a steep mountain, the movement will be greater than a situation where the slope is small. Students should also consider the case where there is some friction and the movement due to inertia is only a fraction of the previous movement. These generalizations are part of building new improved models that take into account more details and better predict what will happen in the real world.

Discussion

One of the most powerful thinking tools that the experts use for Mathematical Modeling is the Potential function. Here we have shown that by using boards this tool can be taught to elementary school students, starting from first grade. Moreover, according to our empirical experience with thousands of students this is a very attractive activity for them. It can be easily included in meaningful and stimulating stories. It can also be adapted to concrete activities with pill bugs, worms, dogs, toy mountains, and bikes. Student can also play on tile floors where they can walk through the solutions.

Figure 7: walking through the solution.

The use of potentials and boards can be progressively continued throughout all K-12 grades. Every year, students can review the models where new number domains, operations, rules, and situations are added. Potentials and boards are also very fertile types of models. They can be used in a wide variety of everyday life situations. They suggest powerful spatial analogies to problems of different nature. They can also provide interesting application for other mathematics contents of the curriculum and be a very effective way to connect them with everyday life.

On the other hand, the idea of introducing the notion of time steps and devising mechanisms to decide the next position based on the present position is a good strategy to start approaching calculus (Marland & Searcy,2010) and differential equations. We have introduced two core mechanisms. One mechanism that searches in the neighbor cells for the best one. The other includes inertia. These are very important core dynamics that are widely used in a great number of situations.

In conclusion, combining the USABle strategy that starts using models, then continue by selecting models, then adjusting models and finally building models, with the strategy of modeling with boards and potentials, teachers have a very feasible and powerful tool to introduce modeling in the elementary school.

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